$n^{\underline{k}} = n(n-1)\cdots(n-k+1)$

$\binom{n}{k} = \dfrac{n^{\underline{k}}}{k!}$

排列組合

	可重覆	不重覆
排列	$U^n_k = n^k$	$P^n_k = n^{\underline{k}}$
組合	$H^n_k = \binom{n+k-1}{k}$	$C^n_k = \binom{n}{k}$

例 $\{a, b, c\}$ $n=3,\ k=2$

$U^3_2 = 9$, aa, ab, ac, ba, bb, bc, ca, cb, cc

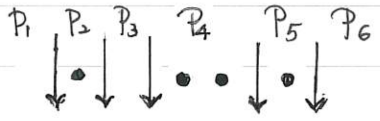
$P^3_2 = 6$, ab, ac, ba, bc, ca, cb

$H^3_2 = 6$, aa, bb, cc, ab, bc, ca

$C^3_2 = 3$, ab, bc, ca

• $H^n_k = \binom{n+k-1}{k}$, $\begin{cases} n = 6 \text{ 人} \\ k = 4 \text{ 球} \end{cases}$

(甲)



2 4 4 5

$\begin{cases} n-1 = 5 & \downarrow \\ k = 4 & \bullet \end{cases}$ (排列) $\dfrac{(5+4)!}{5!\,4!} = \dfrac{(n+k-1)!}{(n-1)!\,k!}$

$H^6_4 = \binom{9}{4} = \binom{n+k-1}{k}$

(乙) $1 \le c_1 \le c_2 \le c_3 \le \underset{(k)}{c_4} \le 6 = n$

$\downarrow +0 \quad \downarrow +1 \quad \downarrow +2 \quad \downarrow +3 = k-1$

$1 \le d_1 < d_2 < d_3 < d_4 \le 9 = n+k-1$

2 5 6 8

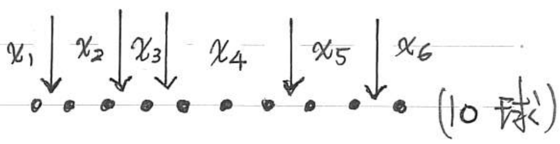
(丙) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = k$ $(x_i \ge 0)$

$0 + 1 + 0 + 2 + 1 + 0 = 4$

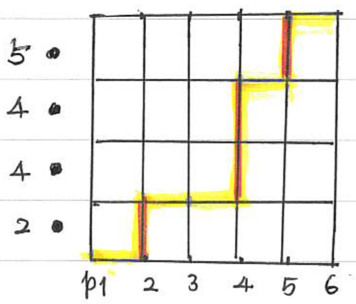
$+1 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$ $\quad y_i = x_i + 1 \downarrow$

$1 + 2 + 1 + 3 + 2 + 1 = 10$

$y_1 + y_2 + y_3 + y_4 + y_5 + \underset{(n)}{y_6} = n + k$ $(y_i > 0)$ $\quad \binom{9}{5} = \binom{n+k-1}{n-1}$



(丁)



$C^9_5 = \dfrac{(5+4)!}{5!\,4!}$ paths

• Recurrence $\quad P_n$ 不選 $\quad$ 選

(1) $C^n_k = C^{n-1}_k + C^{n-1}_{k-1}$

(2) $P^n_k = P^{n-1}_k + k\,P^{n-1}_{k-1}$ $\quad \left(\Delta n^{\underline{k}} = k\,n^{\underline{k-1}}\right)$

(3) $H^n_k = H^{n-1}_k + H^n_{k-1}$

Generating function method

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。組合 (Ordinary generating function : OGF) : $\sum_k a_k z^k$

(1) $\begin{cases} a \\ b \\ c \end{cases}$: $(1+a)\ (1+b)\ (1+c) = 1+(a+b+c)+(ab+bc+ca)+abc$

$(1+az)(1+bz)(1+cz) = 1+(a+b+c)z+(ab+bc+ca)z^2+abcz^3$

$(a=b=c=1) \quad = 1+3z+3z^2+z^3$

$C^n_k = \binom{n}{k}$: $(1+z)\cdots(1+z) = \sum_k C^n_k z^k$, $\quad (1+z)^n = \sum_k \binom{n}{k} z^k$ (Binomial th)

(2) $\begin{cases} aaa \\ b \\ c \end{cases}$: $(1+az+a^2z^2+a^3z^3)(1+bz)(1+cz) = 1+(a+b+c)z+(a^2+ab+ac+bc)z^2$
$+(a^3+a^2b+a^2c+abc)z^3+(a^3b+a^3c+a^2bc)z^4+a^3bc\,z^5$

(3) $\begin{cases} aaaaa \\ bb \\ ccc \end{cases}$: $(1+az+a^2z^2+a^3z^3+a^4z^4+a^5z^5)(1+bz+b^2z^2)(1+cz+c^2z^2+c^3z^3) = \cdots$

(4) $\begin{cases} a_1,\cdots,a_n \\ \text{重覆選取} \end{cases}$ $(1+z+z^2+\cdots)(1+z+z^2+\cdots)\cdots(1+z+z^2+\cdots) = \sum_k H^n_k z^k$

$\left(\frac{1}{1-z}\right)^n = (1-z)^{-n} = \sum_k \binom{-n}{k}(-z)^k$, $\quad \therefore H^n_k = \binom{-n}{k}(-1)^k = \binom{n+k-1}{k}$

。排列 (Exponential generating function : EGF) : $\sum_k a_k \frac{z^k}{k!}$

(1) $\begin{cases} aaa \\ bb \end{cases}$: $\left(1+\frac{a}{1!}z+\frac{a^2}{2!}z^2+\frac{a^3}{3!}z^3\right)\left(1+\frac{b}{1!}z+\frac{b^2}{2!}z^2\right)$

$= 1+\left(\frac{a}{1!}+\frac{b}{1!}\right)\frac{z}{1!}+\left(\frac{a^2}{2!}+\frac{a}{1!}\frac{b}{1!}+\frac{b^2}{2!}\right)2!\frac{z^2}{2!}+\left(\frac{a^3}{3!}+\frac{a^2}{2!}\frac{b}{1!}+\frac{a}{1!}\frac{b^2}{2!}\right)3!\frac{z^3}{3!}$
$+\left(\frac{a^3}{3!}\frac{b}{1!}+\frac{a^2}{2!}\frac{b^2}{2!}\right)4!\frac{z^4}{4!}+\frac{a^3}{3!}\frac{b^2}{2!}5!\frac{z^5}{5!}$

(2) $\begin{cases} a_1\cdots a_n \\ \text{重覆排列} \end{cases}$: $\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots\right)^n = \sum_k U^n_k \frac{z^k}{k!}$

$e^{nz} = \sum_k \frac{(nz)^k}{k!}$, $\quad \therefore U^n_k = n^k$

(3) $\{a_1,\cdots,a_n$ 不可重覆 : $\left(1+\frac{z}{1!}\right)^n = \sum_k P^n_k \frac{z^k}{k!}$ $\quad P^n_k = n^{\underline{k}}$

(4取k, 可重覆排列)

(4) $\begin{cases} x_1x_2\cdots x_k \\ x_i = 0,1,2,3 \end{cases}$ 1, 2, 3 至少出現一次 : $\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)^3 = e^z(e^z-1)^3$

$= e^{4z}-3e^{3z}+3e^{2z}-e^z = \sum_k \frac{4^k-3\cdot3^k+3\cdot2^k-1}{k!}z^k$

$\therefore A = 4^k-\binom{3}{1}3^k+\binom{3}{2}2^k-1$

(5) $\begin{cases} x_1x_2\cdots x_k \\ x_i = 0,1,2,3 \end{cases}$ 0 出現偶數次 : (甲) $\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\left(1+z+\frac{z^2}{2!}+\cdots\right)^3 = \frac{e^z+e^{-z}}{2}e^{3z} = \frac{e^{4z}+e^{2z}}{2}$

$= \sum_k \frac{4^k+2^k}{2}\frac{z^k}{k!}$

$\therefore B = \frac{1}{2}(4^k+2^k)$

(乙) $4^k \begin{cases} 2^k : \text{不含 } 0/1 \text{ (只含 } 2,3) \\ 4^k-2^k : \text{含 } 0/1 \text{ (奇偶各半)} \end{cases}$ $\quad \therefore B = 2^k+\frac{1}{2}(4^k-2^k)$

$\times\times\times\times\,{}^{0}_{1}\,232$

Binomial Coefficients

$$\binom{r}{k} = \begin{cases} \dfrac{r^{\underline{k}}}{k!} = \dfrac{r(r-1)\cdots(r-k+1)}{k!} & \text{integer } k \geq 0 \\ 0 & k < 0 \end{cases}$$

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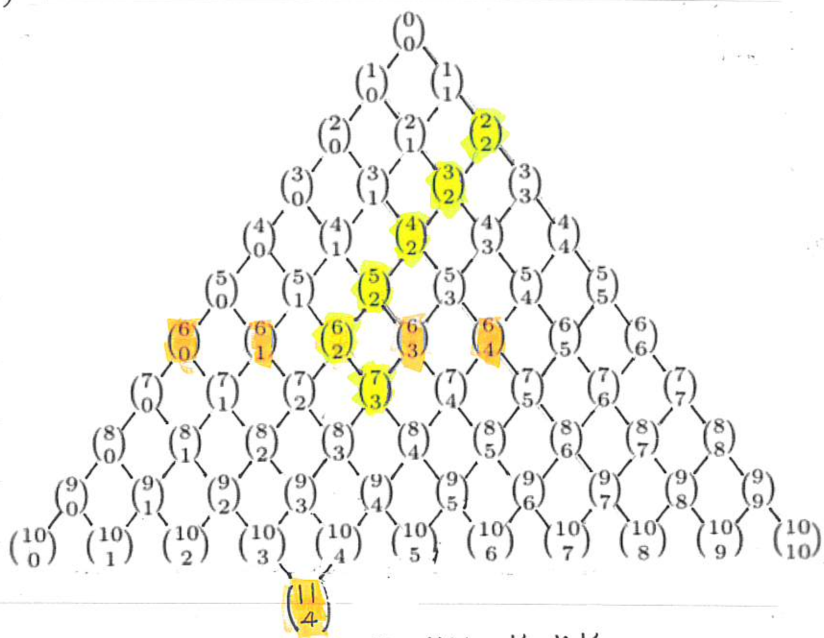
Table 174 The top ten binomial coefficient identities.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad (0 \leq k \leq n)$$

(甲)組合意義 (乙)Pascal三角形



$$\binom{n}{k} = \binom{n}{n-k},$$

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1},$$

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

$$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$$

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k},$$

$$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r,$$

$$(x+y)(x+y)\cdots(x+y) = \sum_k \binom{r}{k} x^k y^{r-k}$$

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n},$$

$$\binom{2}{0}+\binom{3}{1}+\binom{4}{2}+\binom{5}{3}+\binom{6}{4} = \binom{7}{4}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1},$$

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2} = \binom{7}{3}$$

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n},$$

$$\binom{6}{0}\binom{5}{4}+\binom{6}{1}\binom{5}{3}+\binom{6}{2}\binom{5}{2}+\binom{6}{3}\binom{5}{1}+\binom{6}{4}\binom{5}{0} = \binom{11}{4}$$

(Vandermonde convolution) $\quad (1+z)^6(1+z)^5 = (1+z)^{11}$

Prob 1
$$\sum_{0 \leq k \leq m} \frac{\binom{m}{k}}{\binom{n}{k}} = \sum_{0 \leq k \leq m} \frac{\binom{n-k}{m-k}}{\binom{n}{m}} = \frac{1}{\binom{n}{m}}\left[\binom{n}{m}+\binom{n-1}{m-1}+\cdots+\binom{n-m}{0}\right] = \frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{n+1}{n-m+1}$$

Prob 2
$$\sum_{0 \leq k \leq n} \frac{k\binom{m-k-1}{m-n-1}}{\binom{m}{n}} \quad (m > n \geq 0) \quad \text{(Sorting literature)}$$

$$= \frac{1}{\binom{m}{n}} \sum_{0 \leq k \leq n} m\binom{m-k-1}{m-n-1} - \frac{1}{\binom{m}{n}} \sum_{0 \leq k \leq n} (m-k)\binom{m-k-1}{m-n-1}$$

$$= \frac{m}{\binom{m}{n}}\left[\binom{m-1}{m-n-1}+\binom{m-2}{m-n-1}+\cdots+\binom{m-n-1}{m-n-1}\right] - \frac{m-n}{\binom{m}{n}} \sum_{0 \leq k \leq n} \binom{m-k}{m-n}$$

$$= \frac{m}{\binom{m}{n}}\binom{m}{m-n} - \frac{m-n}{\binom{m}{n}}\left[\binom{m}{m-n}+\binom{m-1}{m-n}+\cdots+\binom{m-n}{m-n}\right]$$

$$= m - \frac{m-n}{\binom{m}{n}}\binom{m+1}{m-n+1}$$

$$= \frac{n}{m-n+1}$$

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(Vandermonde convolution $\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \Rightarrow$ 5.22 ~ 5.26)

Table 169 Sums of products of binomial coefficients.

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, \quad \text{integers } m, n. \quad (5.22)$$

$(k \leftarrow k-m)$

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}, \quad \text{integer } l \geqslant 0, \text{ integers } m, n. \quad (5.23)$$

$\binom{l}{m+k} = \binom{l}{l-m-k}$

$$\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}, \quad \text{integer } l \geqslant 0, \text{ integers } m, n. \quad (5.24)$$

$$\sum_{k \leqslant l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}, \quad \text{integers } l, m, n \geqslant 0. \quad (5.25)$$

$$\sum_{0 \leqslant k \leqslant l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \quad \text{integers } l, m \geqslant 0, \text{ integers } n \geqslant q \geqslant 0. \quad (5.26)$$

Prob 4 $\displaystyle \sum_{0 \le k \le n} \frac{k\binom{m-k-1}{m-n-1}}{\binom{m}{n}} = \frac{1}{\binom{m}{n}} \sum_{0 \le k \le \cancel{n}\, m-1} \binom{k}{1}\binom{m-1-k}{m-1-n} \xrightarrow[m \leftarrow m-1-n,\ n \leftarrow 1]{l \leftarrow m-1,\quad q \leftarrow 0} \frac{1}{\binom{m}{n}}\binom{m}{m-\cancel{n+1}\ n-1} = \frac{n}{m-n+1}$

$(m > n \geq 0)$ (5.26)

Prob 5 $\displaystyle \sum_k \binom{n}{k}\binom{s}{k}k = \sum_k \binom{n}{k}\binom{s-1}{k-1}s \xrightarrow[m \leftarrow 0,\quad n \leftarrow -1]{l \leftarrow n,\quad s \leftarrow s-1} s\binom{n+s-1}{n-1}$

(integer $n \geq 0$) (5,23)

(5,23) $\begin{cases} l \leftarrow n+1, & s \leftarrow -n-1 \\ m \leftarrow 1, & n \leftarrow 0 \end{cases}$

Prob 6 $\displaystyle \sum_k \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \sum_k \binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1} = \sum_k \binom{n+k}{k}\binom{n+1}{k+1}\frac{(-1)^k}{n+1}$

(integer $n \geq 0$) $\displaystyle = \frac{1}{n+1}\sum_k \binom{-n-1}{k}\binom{n+1}{k+1} = \frac{1}{n+1}\binom{0}{n} = \delta_{n=0}$

- $\displaystyle \sum_{k \le m} (-1)^k \binom{r}{k} = \binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^m\binom{r}{m} = (-1)^m\binom{r-1}{m}$ (integer m) (5.16)

(從$\{1,2,\ldots,7\}$選取 $\le m=4$個)

$r=7,\ m=4,\quad \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} = \binom{6}{4}$ = 偶數−奇數選法

$m=5,\quad \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} = -\binom{6}{5}$

$m=7,\quad \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} + \binom{7}{6} - \binom{7}{7} = -\binom{6}{7} = 0$

証 (甲) Induction: $(m=4) \Leftrightarrow (m=5)$, $(m=7)$: $\sum_k \binom{7}{k}x^k = (1+x)^7$ $(x=-1)$

(乙) $\displaystyle \sum_{k \le m} (-1)^k\binom{r}{k} = \sum_{k \le m}\binom{k-r-1}{k} = \binom{-r+m}{m} = (-1)^m\binom{r-1}{m}$

(丙) $(m=7)$ $\begin{cases} abcd \\ abcd7 \end{cases}$ (奇偶各半)

$(m=4)$ $\begin{cases} 不含7 = \{\varepsilon, a, ab, abc\} \cup \{abcd\} \\ 含7 = \underbrace{\{7, a7, ab7, abc7\}}_{(奇偶各半)} \end{cases}$ 選取 < 4個 　取4個 $\binom{6}{4}$

$$\begin{cases} F = 1 + r_1 + r_1 r_2 + r_1 r_2 r_3 + \cdots \\ aF = a + a r_1 + a r_1 r_2 + a r_1 r_2 r_3 + \cdots \end{cases}$$

Hypergeometric functions

$$\bullet\ F\left(\left.\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_m \end{matrix}\right| z\right) = \sum_{k\ge 0} \frac{a_1^{\overline{k}} a_2^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} b_2^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{z^k}{k!} = \sum_{k\ge 0} t_k \Leftrightarrow \begin{cases} k \geq 0 \\ t_0 = 1 \\ \dfrac{t_{k+1}}{t_k} = \dfrac{(k+a_1)\cdots(k+a_m)}{(k+b_1)\cdots(k+b_n)} \dfrac{z}{(k+1)} \end{cases}$$

$(k=2)$

$$\bullet\ F\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| z\right) = 1 + \frac{a \cdot b}{c} z + \frac{a(a+1)\, b(b+1)}{c(c+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)\, b(b+1)(b+2)}{c(c+1)(c+2)} \frac{z^3}{3!} + \cdots$$

註: (1) $a_k = b_k$ 可消去, (2) $t_0 = a$ 可提出, (3) $\dfrac{t_{k+1}}{t_k} = \dfrac{k^2+7k+10}{4k^2+1} = \dfrac{(k+2)(k+5)(k+1)(\frac{1}{4})}{(k+\frac{i}{2})(k-\frac{i}{2})(k+1)}$

$$\Rightarrow t_0 F\left(\left.\begin{matrix} 2, 5, 1 \\ \frac{i}{2}, -\frac{i}{2} \end{matrix}\right| \frac{1}{4}\right)$$

Hypergeometric functions:

(1) $F\left(\left.\begin{matrix} 1 \\ 1 \end{matrix}\right| z\right) = F\left(\left.\begin{matrix} \\ \end{matrix}\right| z\right) = \sum_{k\ge 0} \frac{z^k}{k!} = e^z$

(2) $F\left(\left.\begin{matrix} 1, a \\ 1 \end{matrix}\right| z\right) = F\left(\left.\begin{matrix} a \\ \end{matrix}\right| z\right) = \sum_{k\ge 0} \frac{a^{\overline{k}} z^k}{k!} = (1-z)^{-a}$ $\quad (-1)^k a^{\overline{k}} = (-a)^{\underline{k}} = (-a)(-a-1)\cdots(-a-k+1)$

(3) $F\left(\left.\begin{matrix} 1 \\ 1, b \end{matrix}\right| z\right) = F\left(\left.\begin{matrix} \\ b \end{matrix}\right| z\right) = \sum_{k\ge 0} \frac{(b-1)!}{(b-1+k)!} \frac{z^k}{k!} = I_{b-1}(2\sqrt{z}) \frac{(b-1)!}{z^{\frac{b-1}{2}}}$ (Modified Bessel)

(4) $F\left(\left.\begin{matrix} a \\ b \end{matrix}\right| z\right) = \sum_{k\ge 0} \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{z^k}{k!} = M(a, b, c)$ (Confluent Hypergeometric Series)

(5) $F\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| 1\right) = \dfrac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)}$ $\begin{cases} \text{integer } b \leq 0 \\ \mathrm{Re}(c) > \mathrm{Re}(a) + \mathrm{Re}(b) \end{cases}$ (Gaussian Hypergeometric)

$$\Leftrightarrow F\left(\left.\begin{matrix} a, -n \\ c \end{matrix}\right| 1\right) = \frac{(a-c)^{\underline{n}}}{(-c)^{\underline{n}}} = \frac{(c-a)^{\overline{n}}}{c^{\overline{n}}}$$

$$\Leftrightarrow \binom{r+s}{n} = \sum_{k\ge 0} \binom{r}{k}\binom{s}{n-k} \quad \left[= \binom{s}{n} F\left(\left.\begin{matrix} -r, -n \\ s-n+1 \end{matrix}\right| 1\right), \begin{cases} t_0 = \binom{s}{n} \\ \dfrac{t_{k+1}}{t_k} = \dfrac{\binom{r}{k+1}\binom{s}{n-k-1}}{\binom{r}{k}\binom{s}{n-k}} = \dfrac{(r-k)(n-k)}{(k+1)(s-n+k+1)} \end{cases}\right]$$

證 $F\left(\left.\begin{matrix} -r, -n \\ s-n+1 \end{matrix}\right| 1\right) = \dfrac{(r+s)^{\underline{n}}}{s^{\underline{n}}}$ $\begin{cases} r \leftarrow -a \\ s \leftarrow c+n-1 \end{cases}$

$$\therefore F\left(\left.\begin{matrix} a, -n \\ c \end{matrix}\right| 1\right) = \frac{(c-a+n-1)^{\underline{n}}}{(c+n-1)^{\underline{n}}} = \frac{(c-a)^{\overline{n}}}{c^{\overline{n}}}$$

(6) $F\left(\left.\begin{matrix} a, b \\ 1+b-a \end{matrix}\right| -1\right) = \dfrac{(b/2)!}{b!} (b-a)^{\underline{b/2}}$ (Kummer)

(7) $F\left(\left.\begin{matrix} a, b, c \\ 1+c-a, 1+c-b \end{matrix}\right| 1\right) = \cdot\cdot$ (Dixon)

(8) $F\left(\left.\begin{matrix} a, b, -n \\ c, a+b-c-n+1 \end{matrix}\right| 1\right) =$ (Saalschütz identity)

Hypergeometric Transformations

(1) $\dfrac{1}{(1-z)^a} F\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| \dfrac{-z}{1-z}\right) = F\left(\left.\begin{matrix} a, c-b \\ c \end{matrix}\right| z\right)$ (Pfaff reflection)

(2) $F\left(\left.\begin{matrix} a, -n \\ c \end{matrix}\right| z\right) = \dfrac{(a-c)^{\underline{n}}}{(-c)^{\underline{n}}} F\left(\left.\begin{matrix} a, -n \\ 1-n+a-c \end{matrix}\right| 1-z\right)$

(3) $\dfrac{d}{dz} F\left(\left.\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix}\right| z\right) = \dfrac{a_1 \cdots a_m}{b_1 \cdots b_m} F\left(\left.\begin{matrix} a_1+1, \ldots, a_m+1 \\ b_1+1, \ldots, b_n+1 \end{matrix}\right| z\right)$

(4) $F\left(\left.\begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix}\right| z\right) = F\left(\left.\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}\right| 4z(1-z)\right)$

$$F\left(\begin{matrix}a,-n\\c\end{matrix}\middle|\,1\right)=\frac{(a-c)^{\underline{n}}}{(-c)^{\underline{n}}}$$

- **Example 1** $\displaystyle\sum_{0\le k\le n}\binom{r+k}{k}=\binom{r+n+1}{n}$ $(k\le n)$

$$\begin{cases} t_0=\binom{r+n}{n}\\[4pt] \dfrac{t_{k+1}}{t_k}=\dfrac{\binom{r+n-k-1}{n-k-1}}{\binom{r+n-k}{n-k}}=\dfrac{(n-k)\quad(k+1)\cdot 1}{(r+n-k)\,(k+1)}\end{cases}$$

$k\leftarrow n-k$

$$=\sum_{k\ge0}\binom{r+n-k}{n-k}=\binom{r+n}{n}F\left(\begin{matrix}1,-n\\-n-r\end{matrix}\middle|\,1\right)=\binom{r+n}{n}\frac{(r+n+1)^{\underline{n}}}{(r+n)^{\underline{n}}}=\binom{r+n+1}{n}$$

- **Example 2** $\displaystyle\sum_{0\le k\le m}\frac{\binom{m}{k}}{\binom{n}{k}}=\frac{n+1}{n-m+1}$ $(0\le k)$

$$\begin{cases} t_0=1\\[4pt] \dfrac{t_{k+1}}{t_k}=\dfrac{\binom{m}{k+1}\binom{n}{k}}{\binom{n}{k+1}\binom{m}{k}}=\dfrac{(m-k)\,(k+1)\cdot 1}{(n-k)\,(k+1)}\end{cases}$$

$$=F\left(\begin{matrix}1,-m\\-n\end{matrix}\middle|\,1\right)=\frac{(n+1)^{\underline{m}}}{n^{\underline{m}}}=\frac{n+1}{n-m+1}$$

- **Example 3** $(m>n\ge0)$ $\displaystyle\sum_{0\le k\le n}\frac{k\binom{m-k-1}{m-n-1}}{\binom{m}{n}}=\frac{n}{m-n+1}$ $(1\le k)$ $(\because t_0=0)$

$$\begin{cases} t_0=\binom{m-2}{m-n-1}\Big/\binom{m}{n}=\binom{m-2}{n-1}\Big/\binom{m}{n}\\[4pt] \dfrac{t_{k+1}}{t_k}=\dfrac{(k+2)\binom{m-k-3}{m-n-1}}{(k+1)\binom{m-k-2}{m-n-1}}=\dfrac{(k+2)(n-k-1)\cdot 1}{(k+1)(m-k-2)}\end{cases}$$

$k\leftarrow k+1$

$$=\sum_{k\ge0}\frac{(k+1)\binom{m-k-2}{m-n-1}}{\binom{m}{n}}=\frac{\binom{m-2}{n-1}}{\binom{m}{n}}F\left(\begin{matrix}2,1-n\\2-m\end{matrix}\middle|\,1\right)=\frac{(m-2)^{\underline{n-1}}\;n\cdot m^{\underline{n-1}}}{m^{\underline{n}}\;(m-2)^{\underline{n-1}}}=\frac{n}{m-n+1}$$

- **Example 4** $\displaystyle\sum_{k\ge0}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1}=\delta_{n=0}$

$$\begin{cases} t_0=1\\[4pt] \dfrac{t_{k+1}}{t_k}=\dfrac{\binom{n+k+1}{2k+2}\binom{2k+2}{k+1}\frac{(-1)^{k+2}}{(k+2)}}{\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^{k+1}}{(k+1)}}\end{cases}$$

$$=\frac{(n+k+1)(n-k)(2k+2)(2k+1)(k+1)(-1)}{(2k+2)(2k+1)(k+1)(k+1)(k+2)}=\frac{(n+k+1)(k-n)\cdot1}{(k+2)(k+1)}$$

$$=F\left(\begin{matrix}n+1,-n\\2\end{matrix}\middle|\,1\right)=\frac{(n-1)^{\underline{n}}}{(-2)^{\underline{n}}}=\delta_{n=0}$$

- **Example 5** $\displaystyle\sum_{k\le m}(-1)^k\binom{r}{k}=(-1)^m\binom{r-1}{m}$

$$\begin{cases} t_0=(-1)^m\binom{r}{m}\\[4pt] \dfrac{t_{k+1}}{t_k}=\dfrac{(-1)\binom{r}{m-k-1}}{\binom{r}{m-k}}=\dfrac{(-1)(m-k)\quad(k+1)}{(r-m+k+1)\,(k+1)}\end{cases}$$

$k\leftarrow m-k$

$$=\sum_{k\ge0}(-1)^{m-k}\binom{r}{m-k}$$

$$=(-1)^m\binom{r}{m}F\left(\begin{matrix}1,-m\\r-m+1\end{matrix}\middle|\,1\right)=(-1)^m\binom{r}{m}\frac{(m-r)^{\underline{m}}}{(m-r-1)^{\underline{m}}}=(-1)^m\binom{r-1}{m}$$

- **Gamma 函數**: $\displaystyle\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt=\lim_{n\to\infty}\frac{n!}{z^{\overline{n+1}}}n^z$

(1) $\Gamma(1)=1$ $\Rightarrow$ $\Gamma(n)=(n-1)!$

(2) $\Gamma(z+1)=z\Gamma(z)$

(3) $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$ $\Rightarrow$ $\Gamma\left(\frac12\right)=\sqrt{\pi}$

Partial Hypergeometric Sums

No.
Date: / /

• **定理** $\sum_{a\le k<b} t_k = \sum_a^b t_k \Delta k = T(b)-T(a) \iff \Delta(T_k) = T_{k+1}-T_k = t_k$

• **定理 (Gosper)** $\begin{cases} (1)\ \dfrac{t_{k+1}}{t_k} = \dfrac{p_{k+1}\, q_k}{p_k\ r_{k+1}} & \left(\dfrac{t_{k+1} r_{k+1}}{p_{k+1}} = \dfrac{t_k q_k}{p_k}\right) \\ (2)\ S_k \text{ 滿足 } p_k = q_k S_{k+1} - r_k S_k \end{cases} \Rightarrow (3)\ T_k = \dfrac{t_k r_k S_k}{p_k}$ 滿足 $\Delta(T_k) = t_k$

証 $T_{k+1}-T_k = \dfrac{t_{k+1} r_{k+1} S_{k+1}}{p_{k+1}} - \dfrac{t_k r_k S_k}{p_k} = \dfrac{t_k q_k S_{k+1}}{p_k} - \dfrac{t_k r_k S_k}{p_k} = t_k$ #

• **解** $p_k = q_k S_{k+1} - r_k S_k$

(a) $\dfrac{t_{k+1}}{t_k} = \dfrac{(k+a_1)\cdots(k+a_m)z}{(k+b_1)\cdots(k+b_n)(k+1)} = \dfrac{p_{k+1}\, q_k}{p_k\ r_{k+1}}$, $\begin{cases} p_k \equiv 1, & q_k = (k+a_1)\cdots(k+a_m)z \\ & r_k = (k-1+b_1)\cdots(k-1+b_n)k \end{cases}$

(b) $\begin{cases} k+\alpha \mid q_k \\ k+\beta \mid r_k \\ (k+\beta+1 \mid r_{k+1}) \end{cases} \Rightarrow (\alpha-\beta) \notin \mathbb{N}$, 否則 $\dfrac{p_{k+1}(k+\alpha)q_k'}{p_k(k+\beta+1)r_{k+1}'} = \dfrac{\overbrace{p_{k+1}(k+\alpha)(k+\alpha-1)\cdots(k+\beta+2)}^{p_{k+1}'}}{\underbrace{p_k\ (k+\alpha-1)\cdots(k+\beta+2)(k+\beta+1)}_{p_k'}} \dfrac{q_k'}{r_{k+1}'}$

(c) (b) $\Rightarrow$ S_k: 多項式, $S_k = ak^d + bk^{d-1} + \cdots$

(d) $a, d = ?$

(例: $1 = (k-n)S_{k+1} - kS_k = (k-n)\left[a(k+1)^d + b(k+1)^{d-1}+\cdots\right] - k\left[ak^d + bk^{d-1}+\cdots\right]$
$= (ad-na)k^d + \cdots$
$\Rightarrow d=0,\ S_k = a = \frac{-1}{n}$)

$2p_k = (q_k - r_k)(S_{k+1}+S_k) + (q_k+r_k)(S_{k+1}-S_k)$
$= \quad Q \quad (2ak^d+\cdots) + \quad R \quad (adk^{d-1}+\cdots)$

$\begin{cases} \deg Q \geqq \deg R & : \deg p_k = \deg Q + d \\ \deg Q < \deg R & \begin{cases} 2\beta + \gamma d \ne 0 : & \deg p_k = \deg R + d - 1 \\ \quad " \quad\ = 0 : & < \end{cases} \end{cases}$ $\quad \begin{cases} Q = \beta k^{d'} + \cdots \\ R = \gamma k^{d'} + \cdots \end{cases}$

• **Example** $\sum_{0\le k\le m} (-1)^k \binom{n}{k} \overset{?}{=} (-1)^m \binom{n-1}{m}$

証 (1) $\dfrac{t_{k+1}}{t_k} = \dfrac{(-1)^{k+1}\binom{n}{k+1}}{(-1)^k\binom{n}{k}} = \dfrac{k-n}{k+1}$ $\quad \begin{cases} p_k \equiv 1 \\ q_k = k-n \\ r_k = k \end{cases}$ $\begin{cases} Q = -n, & \beta = -n, & \ne 0, & 0 = 1+d-1 \\ R = 2k-n, & \gamma = 2, & = 0, & d = n \end{cases}$

$2\beta + \gamma d = -2n + 2d$

(2) $1 = (k-n)S_{k+1} - kS_k \Longrightarrow S_k = \frac{-1}{n}$

(3) $T_k = \dfrac{t_k r_k S_k}{p_k} = \dfrac{(-1)^k\binom{n}{k}k\left(\frac{-1}{n}\right)}{1} = (-1)^{k-1}\binom{n-1}{k-1}$

$\therefore$ 原式 $= (-1)^{k-1}\binom{n-1}{k-1}\Big|_0^{m+1} = (-1)^m\binom{n-1}{m}$ #

驗算

$\Delta T_k = T_{k+1} - T_k$
$= (-1)^k\binom{n-1}{k} - (-1)^{k-1}\binom{n-1}{k-1}$
$= (-1)^k\binom{n}{k}$
$= t_k$